SOLVING GLOBAL OPTIMIZATION
USING NON-PARAMETERIC FILLED FUNCTION

Ismail Bin Mohd1; Goh Khang Wen2; Yosza Bin Dasril3

1Department of Mathematics, Faculty Science and Technology, Universiti Malaysia Terengganu (UMT).
ismailmd@umt.edu.my

2Department of Physical and Mathematical Sciences, Faculty of Science, Engineering and Technology, Universiti Tunku Abdul Rahman (UTAR)
goh_khangwen@yahoo.com

3Department of Industrial Electronics, Faculty Electronics and Computer Engineering, Universiti Teknikal Malaysia Melaka (UTeM).
yosza@utem.edu.my

ABSTRACT

The filled function method is an approach which constructs an auxiliary function to leave from a current local minimizer to a lower minimizer until the global minimum of multimodal function been found. However, all of the filled functions previously proposed in literature are required at least one or two parameters which must be chosen to satisfy several specific conditions. In this paper, we propose a new class of filled function which does not require any parameter (to be selected) for finding the global minimizer. The idea of obtaining the non-parameteric filled function is how to use the current local minimizer to obtain a new local minimizer. For the time being, the idea of integration and combining its with the current local minimizer can be used as a better approach for obtaining the global minimizer as shown in this paper.

Keywords: local and global optimization, nonparametric filled function

ABSTRACT

Metode filled function adalah pendekatan yang membangun fungsi pembantu untuk berpindah dari sebuah minimizer lokal saat ini ke minimizer yang lebih rendah sehingga nilai minimum global dari sebuah fungsi multimodal dapat ditemukan. Akan tetapi, semua filled function yang sebelumnya diajukan dalam kepustakaan membutuhkan paling tidak satu atau dua parameter yang harus dipilih untuk memenuhi beberapa persyaratan khusus. Dalam makalah ini, kami mengusulkan sebuah kelas filled function baru yang tidak membutuhkan parameter apapun (untuk dipilih) untuk mendapatkan minimizer global. Ide untuk memperoleh filled function non-parametrik adalah dengan menggunakan minimizer lokal yang ada untuk mendapatkan sebuah minimizer lokal baru. Untuk sementara ini, ide untuk integrasi dan kombinasi dengan minimizer lokal yang ada dapat digunakan sebagai pendekatan yang lebih baik untuk mendapatkan minimizer global seperti dalam makalah ini.

Kata kunci: optimisasi lokal dan global, filled function non parametrik.
INTRODUCTION

There are several optimization methods had been proposed for solving the constrained and unconstrained optimization problems. The methods such as Newton’s method and quasi-Newton method are well performed for determining a local solution and once say globally determined when solving convex optimization problems (Boyd and Vandenberghe, 2004). But, if the mathematical models contain non-convexity structure, then those methods fail to obtain the global solution since these methods do not know how to pass a hill (for minimization) or cross a valley (for maximization) to locate another better optimum solution (Liu and Teo, 1999).

The filled function method introduced by Ge (1990) is constructed by moving from one isolated local minimizer to another better local minimizer until the global minimizer is found. Subsequently, several classes of filled functions have been proposed by various authors as seeing in (Xian, 2001a; Xian, 2001b; Wang et. al., 2006; Wang and Zhou, 2006; Liang, 2007; Wang et. al., 2007; Shang, 2008). Then, a series of two-parameter and one-parameter filled functions have been proposed in the literature. However, there are some conditions to be satisfied and off course more times are needed before the parameter can be selected (Goh, 2009), and most of the parameters have to be iteratively updated.

In this paper, we propose a new class of filled function which does not require any parameter to be selected for finding global minimizer as mentioned above. This paper is organized as follows. In Section 3, some of literature review of the various filled functions will be presented. The idea of non-parameter filled function which involved the integration and the theoretical prove of our method will be given in Section 4 with the basic properties are written in Section 5. Section 6 contains several numerical results which illustrate our method. Finally, the conclusion which ends this paper is written in Section 7.

The Idea for New Technique

By observing the Fig. 1 and (Ismail, 2008), we would like to reach the point $P_4$ the global minimizer of $f$ by using one of the familiar unconstrained optimization techniques. It is very easy to obtain the local minimizer $P_2$ by using the local search method starting at $P_1$ and also to obtain the local minimizer $P_4$ by using any local search method starting at $P_5$. But when you have found the local minimizer $P_4$, you need a bridge or any means to use the starting point $P_1$ for obtaining the another local minimizer $P_4$ where the function value at $P_2$ and $P_4$ are equal. However, if we can replace $f(x)$ with $P(x, x'_1)$ where $P(x, x'_1)$ is a function obtained by using some transformation of $f$ at $x'_1$ as shown in Fig. 1, then the moving from $P_2$ to $P_3$ can be done using the steepest descent method, say, from $Q_1$ to $Q_2$ whence the mentioned obstacle can be handled. If $P_4$ does not represent the global minimizer, then we can start the process at $X'_2$ for building new $P(x, x'_2)$.
The method produced from the above idea will be called as non-parameter filled function method where its detail explanation will be given in Section 4. Before we proceed with non-parameter filled function method, we will give some overview and rather detailed discussion about the class of parameter filled function methods in Section 3.

Class of Parameter Filled Function

Ge (1990) proposed a two-parameter filled function of $f : \Omega \subset R^n \rightarrow R$ at an isolated local minimizer $x_k^*$ of $f$ over $\Omega$, defined by

$$P(x, x_k^*, r, \rho) = \frac{1}{r + f(x)} \exp \left( -\frac{\|x - x_k^*\|^2}{\rho^2} \right)$$

(3.1)

where the parameters $r$ and $\rho$ need to be chosen appropriately. Generally, the filled function $P$ of the objective function $f$ with the current minimizer $x_k^*$ must satisfy the following conditions.

**Condition 3.1:** $x_k^*$ becomes a strict local maximizer of $P$ and the whole basin $B_k$ will be a part hill of $P$. ♦

**Condition 3.2:** The fill function $P$ has no saddle point or minimizer in any higher basin $B_h$. ♦

**Condition 3.3:** If there exists any basin $B_l$ lower than the current basin $B_k$, then there will exists a point $x_{kB} \in B_k$ that minimizes $P$ through the line $x$ to $x_k^*$. ♦

In fact, this filled function (3.1) has an unfavorable property in numerical implementation. The changes in both the filled function and its gradient are indistinguishable when the distance between the current iteration point, $x$, and $x_k^*$ is large.

In order to overcome the numerical problem, in (Ge and Qin, 1987) they proposed seven other filled functions as follows.

$$\overline{P}(x, x_k^*, r, \rho) = \frac{1}{r + f(x)} \exp \left( -\frac{\|x - x_k^*\|^2}{\rho^2} \right),$$

(3.2)

$$G(x, x_k^*, r, \rho) = -\rho^2 \ln[r + f(x)] - \|x - x_k^*\|^2,$$

(3.3)

$$\overline{G}(x, x_k^*, r, \rho) = -\rho^2 \ln[r + f(x)] - \|x - x_k^*\|,$$

(3.4)

$$Q(x, x_k^*, A) = -[f(x) - f(x_k^*)] \exp \left( A\|x - x_k^*\|^2 \right),$$

(3.5)

$$\overline{Q}(x, x_k^*, A) = -[f(x) - f(x_k^*)] \exp \left( A\|x - x_k^*\|^2 \right),$$

(3.6)

$$\nabla E(x, x_k^*, A) = -\nabla f(x) - 2A[f(x) - f(x_k^*)](x - x_k^*),$$

(3.7)

and
They believed that the last four equations, (3.5) – (3.8), were better choices for filled functions because those are only requiring one parameter, $A$. Nevertheless, In (Wu, 1999), the author proved that $\nabla E$ and $\nabla \overline{E}$ did not exist for general objective function $f(x)$ even for $x \in R^2$. It is also evident that $Q$ and $\overline{Q}$ are still suffer from the same numerical problem as (3.1).

Therefore, several classes of filled functions have been proposed by various authors. In fact, almost all filled functions proposed in or before year, 2000, are variants or generalization of (3.1) and (3.5). More specifically, in (Wang and Sheng 1992), a variant of (3.1) has been suggested by

$$P(x, x^*_k, r, \rho) = \ln \left(1 + \frac{1}{r + f(x)}\right) \exp \left(-\frac{\|x - x^*_k\|^2}{\rho^2}\right)$$

(3.9)

The family of filled functions of (3.1) and (3.9) mentioned in (Zhuang 1994) is given by

$$P(x, x^*_k, r, \rho) = G(r + f(x)) \exp \left(-\frac{\|x - x^*_k\|^2}{\rho^2}\right)$$

(3.10)

where $G : R^+ \to R$ is a convex, twice continuously differentiable, and strictly decreasing function which satisfies

$$[G (y)]^3 \leq G(y) G'(y), \text{ for all } y > 0.$$  

In (Zheng et al., 2001), the function (3.10) had been generalized to give the following family of filled functions

$$P(x, x^*_k, r, \rho) = \phi(r + f(x)) \exp \left(-A \phi \left(\|x - x^*_k\|^{\beta}\right) \right)$$

(3.11)

where $\beta \geq 1$, $A > 0$, and $r$ are chosen such that $r + f(x) > 0$ for all $x \in \Omega$, and the functions $\phi(t)$ and $\omega(t)$ must satisfy the properties

i. $\phi(t)$ and $\omega(t)$ are continuously differentiable for $t \in (0, +\infty)$,

ii. $\phi(t) > 0$, $\phi'(t) < 0$ and $\phi'(t) + \phi(t)$ is monotonically increasing, and

iii. $\phi(0) = 0$, $\omega(t) > 0$ and $\omega(t) \geq c > 0$.

In (Kong, 2000), the author suggested the following variant of (3.5)

$$Q(x, x^*_k, A) = -\ln \left[1 + f(x) - f(x^*_k)\right] \exp \left(A \|x - x^*_k\|^2\right)$$

(3.12)

where $A > 0$ is sufficiently large. In (Zheng et al. 2001) the authors generalized (3.5) and (3.12) by proposing

Solving Global Optimization …… (Ismail Bin Mohd; dkk)
\[ Q(x, x^*_k, A) = -\phi(f(x) - f(x^*_k))\exp(A \|x - x^*_k\|^\beta) \]  \hspace{1cm} (3.13)

where \( \beta \) is any natural number, \( \phi(t) \) is similar to (3.11) and the \( \phi(t) \) must satisfy the properties

\begin{enumerate}
  \item \( \phi(t) \) is continuously differentiable for \( t \in [0, +\infty) \),
  \item \( \phi(0) = 0 \), \( \phi'(t) > 0 \), \( \forall t \geq 0 \), and
  \item \( \phi'(t) - \phi(t) \) is monotonically decreasing for \( t \in (0, +\infty) \).
\end{enumerate}

However, it was discovered (Ng, 2003) that (3.9) - (3.13) still suffer the similar numerical problems as (3.1).

In (You, et al., 2005) the authors proposed a filled function with one parameter as follows.

\[ P(x, x_0, x_1^*, A) = \eta(\|x - x_0\|) - \varphi(A[1 - \exp(-\min\{f(x) - f(x_1^*), 0\})]) \]  \hspace{1cm} (3.14)

where \( A > 0 \), \( x_0 \) is a prefixed point satisfied \( f(x_0) \geq f(x_1^*) \), and \( \eta(t) \) and \( \varphi(t) \) need to satisfy the conditions

\begin{enumerate}
  \item \( \eta(t) \) and \( \varphi(t) \) are strictly monotonically increasing functions for \( t \in [0, +\infty) \),
  \item \( \eta(0) = 0 \) and \( \varphi(0) = 0 \), and
  \item \( \eta(t) \to C > 0 \) as \( x \to +\infty \), where \( C \geq \max_{x \in X_1} \eta(\|x - x_0\|) \) for some \( X_1 \).
\end{enumerate}

There are several other classes of filled functions which have been proposed by various authors (see Xian, 2001a; Xian, 2001b; Wang et. al., 2006; Wang and Zhou, 2006; Liang, 2007; Wang et. al. 2007; Shang, 2008). In fact, most of their proposed filled functions are revised from (3.1) and (3.5). Therefore, a series of two-parameter and one-parameter filled functions have been proposed in the literature. By the way, there are several conditions must be satisfied and take times to be calculated before the parameter can be selected, and most of the parameter has to be iteratively updated.

**Non-Parameteric Filled Function**

In order to obtain a filled function without using any parameter as seen in (3.1) – (3.14) and fulfill the dream of the idea fruited in Section 2, in this paper, for global minimization, we propose a new non-parameter filled function defined by

\[ P(x, x^*) = \omega(x, x^*) + f(x^*)|x - x^*| \]  \hspace{1cm} (4.1)

where

\[ \omega(x, x^*) = \begin{cases} 
- \int_x^{x^*} f(x) \, dx & (x \geq x^*) \\
- \int_{x^*}^x f(x) \, dx & (x < x^*) 
\end{cases} \]

and \( x^* \) is an isolated local minimizer of \( f : \Omega \subset R \to R \). We can write the second expression in (4.1) as
\[ f(x^*)|_{x-x^*} = \begin{cases} \int_{x^*}^{x} f(x^*) \, dx & (x \geq x^*) \\ \int_{x}^{x^*} f(x^*) \, dx & (x < x^*) \end{cases} \quad (4.2) \]

Therefore, the non-parameter filled function (4.1) can be written as

\[ P(x,x^*) = \begin{cases} -\int_{x^*}^{x} (f(x) - f(x^*)) \, dx & (x \geq x^*) \\ -\int_{x}^{x^*} (f(x) - f(x^*)) \, dx & (x < x^*) \end{cases} \quad (4.3) \]

**Example 4.1**

Consider the function \( f : \Omega \subset R \to R \) with \( f(x) = (x-1)^2 - 1 \) for \( x \in [0,2] \). Clearly that \( x^* = 1 \in [0,2] \). The first filled function with \( x^* = 1 \) is given by

\[ P(x,1) = \begin{cases} \frac{(x-1)^3}{3} & (x \geq 1) \\ \frac{(x-1)^3}{3} & (x < 1) \end{cases} \]

**Basic Properties**

We will present several properties which should be satisfied by our new non-parameteric filled function \( P \).

**Theorem 5.1:** If \( x^* \) is an isolated minimizer of \( f : \Omega \subset R \to R \), then \( x^* \) must be a maximizer of \( P(x,x^*) \).

**Proof**

By (4.3), clearly that in the neighborhood of \( x^* \), \( f(x) \geq f(x^*) \). Therefore

\[ P(x,x^*) \leq 0 = P(x^*,x^*) \]

**Theorem 5.2:** If \( x^* \) is an isolated minimizer of \( f : \Omega \subset R \to R \), then \( P(x,x^*) \) does not have any stationary point for \( f(x) > f(x^*) \).

**Proof**

The first derivative of \( P(x,x^*) \) is given by

\[ P'(x,x^*) = \begin{cases} -(f(x) - f(x^*)) & (x \geq x^*) \\ +(f(x) - f(x^*)) & (x < x^*) \end{cases} \quad (5.1) \]
So, for \( f(x) > f(x^*) \), we have

\[
P'(x, x^*) = \begin{cases} 
-ve & (x > x^*) \\
0 & (x = x^*) \\
+ve & (x < x^*) 
\end{cases}
\] (5.2)

which proves the theorem.

**Theorem 5.3:** If \( x^* \) is an isolated minimizer of \( f : \Omega \subset R \to R \), then \( P(x, x^*) \) must have a stationary point for \( f(x) \leq f(x^*) \).

**Proof**

Suppose that there exists a point \( x^{**} \) which is not equal to \( x^* \) such that \( f(x^{**}) = f(x^*) \). Therefore

\[
P'(x^{**}, x^*) = 0
\]

which means that there exists a stationary point for \( f(x) \leq f(x^*) \).

**Testing Examples**

In order to observe the capability of the new idea for solving the global optimization of \( f : \Omega \subset R \to R \) which has been described in Section 4, we have used the following five examples as illustrating where their graphs are given in Figure 6.1 – Figure 6.5 respectively.

<table>
<thead>
<tr>
<th>Examples</th>
<th>Functions</th>
<th>Domains</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( f(x) = \sin(x) + \sin\left(\frac{2}{3}x\right) )</td>
<td>( (0 \leq x \leq 20) )</td>
</tr>
<tr>
<td>2.</td>
<td>( f(x) = \sin\left(\frac{4}{9}x\right) \tan\left(\frac{1}{9}x\right) )</td>
<td>( (-20 \leq x \leq 20) )</td>
</tr>
<tr>
<td>3.</td>
<td>( f(x) = \cos\left(\frac{3}{5}x\right) \cos(2x) + \sin(x) )</td>
<td>( (0 \leq x \leq 12) )</td>
</tr>
<tr>
<td>4.</td>
<td>( f(x) = \cos\left(\frac{2}{5}x\right) \sin\left(\frac{1}{10}x\right) + \cos(x) )</td>
<td>( (-20 \leq x \leq 12) )</td>
</tr>
<tr>
<td>5.</td>
<td>( f(x) = \sin\left(\frac{4}{9}x\right) \sin(x) )</td>
<td>( (-10 \leq x \leq 10) )</td>
</tr>
</tbody>
</table>

Figure 2 Graph of Example 1
Numerical Results

The numerical results for five examples given in Section 6, are shown in Table 7.1.

<table>
<thead>
<tr>
<th>Examples</th>
<th>Initial Point ($x_0$) / Minimizer of filled function</th>
<th>Minimizer of objective function ($x^*$)</th>
<th>Minimum ($f(x^*)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3 15.9845</td>
<td>5.36225</td>
<td>-1.21598</td>
</tr>
<tr>
<td></td>
<td>2 -7.06858 , 7.06858</td>
<td>-14.1372 , 14.1372</td>
<td>-4.02008</td>
</tr>
<tr>
<td>2</td>
<td>7.3151, 5.56982, 10.3374</td>
<td>1.34096, 3.69152, 10.9598</td>
<td>0.352265</td>
</tr>
<tr>
<td>3</td>
<td>1 1</td>
<td>1.34096, 5.94596, 10.9598</td>
<td>-0.888917</td>
</tr>
<tr>
<td>4</td>
<td>1 -3.14159, 3.14159</td>
<td>-4.50953, 4.50953</td>
<td>-0.888917</td>
</tr>
</tbody>
</table>

The bold numbers in Table 7.1 represent the global minimizer of each testing problems.

The explanation of the contents of Table 7.1, is given in Section 8.

DISCUSSION

Let us consider the single variable global optimization problem of Example 1 given by

$$f(x) = \sin(x) + \sin\left(\frac{2}{3}x\right) \quad (0 \leq x \leq 20).$$

By taking $x_0 = 3$ as starting point for local search method, we obtain $x_1^* = 5.36225$ as a local minimizer of $f(x)$ and $f(x_1^*) = -1.21598$. By using this local minimizer, we can construct our first non-parameter filled function $P(x, 5.36225)$ using (4.3).

Furthermore, use $x' = x_1^* + \delta$ and $x'' = x_1^* - \delta$ as the two different initial points to minimize our first non-parameter filled function of $x_1^* = 5.36225$ where $\delta$ is a small real number, from which we use the local search method to obtain the local

$$x_1^{(1)} = 15.9845 \text{ and } x_1^{(2)} = -2.8651$$

which can be used as the starting point from local search in obtaining the local minimization of $f$. Clearly $x_1^{(2)} \not\in \Omega$. Therefore we only can proceed our computation with $x_1^{(1)} = 15.9845$ to give another local minimizer $x_2^* = 17.0392$ of $f(x)$ with $f(x_2^*) = -1.90596$. We repeat the whole process with $x' = x_2^* + \delta$ and $x'' = x_2^* - \delta$ to give
which clearly that both points are not belonged to \( \Omega \). Therefore, we stop the computation here with \( x_2^* = 17.0392 \) as the global minimizer and its global minimum value is \( f(x_2^*) = -1.90596 \).

**CONCLUSION**

In this paper, we have given an overview of the literature reviews on filled functions from which we have found that in all filled functions considered it requires to select at least one parameter and this parameter must satisfy the conditions as mentioned in (Ge, 1990). We are lucky due to the effort of observing lot of papers about filled function methods, God gives us the chance to propose and promote a new class of non-parameter filled function as shown in Section 4 with properties written in Section 5. By this research, we have discovered a method which can be used to bypass the high basin and to overcome the disadvantages highlighted by the researchers in this field, for obtaining the global minimizer of the global optimization problems and in order to show the capability of the method, five examples have been shown in Section 6 as illustrating.

**REFERENCES**


